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Harmonic Functions with Polynomial Growth on Lattice Points

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1. INTRODUCTION

Let f be either an entire function or a real-valued harmonic function in the Euclidean space $\mathbf{R}^n$ $(n \geqslant 2)$. We say that f is of exponential type α, where $0 \leqslant \alpha < \infty$, if

$$\limsup_{r\to\infty} r^{-1} \log \mathscr{M}(f, r) = \alpha,$$

where $\mathscr{M}(f; r)$ is the supremum of $|f|$ over the circle or sphere of radius r centered at the origin of $\mathbf{C}$ (the complex plane) or $\mathbf{R}^n$. Thus, constant functions are of exponential type 0 (by convention, even if the constant is 0). The spaces of all entire functions and all harmonic functions in $\mathbf{R}^n$ of exponential type at most α are denoted by $\mathscr{E}(\alpha)$ and $\mathscr{H}_n(\alpha)$, respectively. The space of all polynomials of degree at most q $(q \geqslant 0)$ in one complex variable is denoted by $\mathscr{P}_q$, and the space of all real-valued polynomials of degree at most q in $\mathbf{R}^n$ is denoted by $\mathscr{P}_{n,q}$. Throughout this note p denotes a positive number, and $[p]$ is the greatest integer not exceeding p.

The following result is due to Valiron [9] (or see, e.g., Boas [4, p. 183]).

THEOREM A. *If $f \in \mathscr{E}(0)$ and*

$$f(z) = O(|z|^p)$$

as $|z| \to \infty$ through integer values, then $f \in \mathscr{P}_{[p]}$.

This theorem fails if f is supposed to be harmonic in the plane, rather than entire. For example, the harmonic function in $\mathbf{C}$,

$$z \rightsquigarrow \operatorname{Im} \sum_{j=0}^{\infty} (j!)^{-2} z^j$$

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is of exponential type 0 (see, e.g., Titchmarsh [8, p. 255]) and vanishes on the real axis. However, we shall show that there are harmonic analogues of Theorem A in which, for example, the function is required to have polynomial growth on two copies of the integers embedded in $\mathbf{C}$. More generally, we shall work in $\mathbf{R}^n$. Our proofs make use of certain results in [3]. I am grateful to the referee for pointing out that some of the results we require can also be found in the papers of Rao [7] and Zeilberger [10].

Before stating our results, we introduce some further notations. An arbitrary point of $\mathbf{R}^n$ is denoted by $X = (x_1, \dots, x_n)$, and we put

$$|X| = (x_1^2 + \cdots + x_n^2)^{1/2}.$$

Throughout this paper m denotes an integer such that $1 \leqslant m \leqslant n-1$, and we put

$$E^m = \{X \in \mathbf{R}^n : x_{m+1} = \cdots = x_n = 0\}$$

and

$$I^m = \{X \in E^m : x_1, \dots, x_m \in \mathbf{Z}\},$$

where $\mathbf{Z}$ is the set of all integers. If G_1 and G_2 are subsets of $\mathbf{R}^n$, we put

$$G_1 + G_2 = \{X + Y : X \in G_1,\ Y \in G_2\}.$$

The Alexandroff point (at infinity) of $\mathbf{R}^n$ is denoted by $\mathscr{A}$ and the origin of $\mathbf{R}^n$ is denoted by O.

The result from which our harmonic analogues of Theorem A will be deduced is as follows.

THEOREM 1. *If $h \in \mathscr{H}_n(0)$ and*

$$h(X) = O(|X|^p)$$

as $X \to \mathscr{A}$ through I^m, then the restriction of h to E^m is a polynomial of degree at most $[p]$ in $x_1, \dots, x_m$.

The result fails if $h \in \mathscr{H}_n(\alpha)$, where $\alpha > 0$.

Using the case $m = n-1$ of this theorem together with certain other results, we shall prove the following theorems.

THEOREM 2. *Let Z be a point of $\mathbf{R}^n \backslash E^{n-1}$. If $h \in \mathscr{H}_n(0)$ and*

$$h(X) = O(|X|^p)$$

as $X \to \mathscr{A}$ through the set $I^{n-1} + \{O, Z\}$, then $h \in \mathscr{P}_{n,[p]+1}$. The maximal degree $[p]+1$ of h is best possible.

THEOREM 3. *Let Z_0 be a point of E^{n-1}. If $h \in \mathscr{H}_n(0)$ and there exist positive numbers p and p' such that*

$$h(X) = O(|X|^p)$$

as $X \to \mathscr{A}$ through the set I^{n-1} and

$$\frac{\partial h}{\partial x_n}(X) = O(|X|^{p'})$$

as $X \to \mathscr{A}$ through the set $I^{n-1} + \{Z_0\}$, then $h \in \mathscr{P}_{n,p^*}$, where $p^* =$ $\max\{[p], [p'] + 1\}$. The value of p^* is best possible.

2. A LEMMA ON INTERPOLATION

To prepare the way for the proof of Theorem 1 in the cases where $2 \leqslant m \leqslant n - 1$, we prove the following.

LEMMA 1. *Let g be a real-valued function on E^m ($n \geqslant 3$, $m \geqslant 2$) such that the restriction of g to any line of the form*

$$\{X \in E^m : x_i = k_i\,, i \neq j\} \qquad (j = 1,..., m;\ k_i \in \mathbf{Z})$$

is a polynomial (in one variable) of degree at most q, where q is a fixed non-negative integer. Then there exists a polynomial P in $\mathbf{R}^n$ such that $P = g$ on I^m.

Let X_j ($j = 1,..., (q + 1)^m$) be the elements of the set

$$\{X \in I^m : 0 \leqslant x_i \leqslant q\ (i = 1,..., m)\}$$

in some order, and let x_{ji} be the ith coordinate of X_j. Define P in $\mathbf{R}^n$ by the equation

$$P(x_1 ,..., x_n) = \sum_{j=1}^{(q+1)^m} \prod_{i=1}^{m} \prod_{\substack{\nu_i=0 \\ \nu_i \neq x_{ji}}}^{q} \left(\frac{x_i - \nu_i}{x_{ji} - \nu_i}\right) g(X_j).$$

Then P is a polynomial in $\mathbf{R}^n$ (depending only on $x_1 ,..., x_m$) and $P(X_j) = g(X_j)$ for each j. We show, by induction on μ, that if $1 \leqslant \mu \leqslant m$, then $P = g$ on the set

$$J^\mu = \{X \in I^m : 0 \leqslant x_i \leqslant q\ (\mu < i \leqslant m)\}.$$

Suppose first that $\mu = 1$ and that $Y \in J^1$. The restriction of P to the line

$$L_1 = \{X \in E^m: x_i = y_i \ (2 \leqslant i \leqslant m)\}$$

is a polynomial of degree at most q in x_1. Further, L_1 contains $q + 1$ of the points X_j. Since $P(X_j) = g(X_j)$, the restriction of $P - g$ to L_1 is a polynomial of degree at most q in x_1 with at least $q + 1$ zeroes. Hence $P = g$ on L_1, and therefore, since Y is arbitrary, on J^1.

Now suppose that $1 \leqslant \mu < n$, that $P = g$ on J^μ and that $Y \in J^{\mu+1}$. The restriction of P to the line

$$L_2 = \{X \in E^m: x_i = y_i \ (1 \leqslant i \leqslant m, i \neq \mu + 1)\}$$

is a polynomial of degree at most q in $x_{\mu+1}$. Further, L_2 contains $q + 1$ points of J^μ. Since $P = g$ on J^μ, the restriction of $P - g$ to L_2 is a polynomial of degree at most q in $x_{\mu+1}$ with at least $q + 1$ zeroes. Hence $P = g$ on L_1, and therefore, since Y is arbitrary, on $J^{\mu+1}$. The induction is complete.

3. Proof of Theorem 1

We start with the case $m = 1$. To prove this case, we recall an earlier result. Writing $D^j = \partial^j/\partial x_1{}^j$ $(j = 1, 2,\ldots)$, we have the following lemma [3, Theorem 6].

Lemma A. *If h is harmonic in $\mathbf{R}^n$ and*

$$\mathscr{M}(h; r) = O(e^{\beta r}) \qquad (r \to \infty),$$

where $\beta > 0$, then

$$D^j h(O) = O(j^{n-3/2}\beta^j) \qquad (j \to \infty).$$

Suppose that h satisfies the hypotheses of Theorem 1 with $m = 1$. Define a function f: $\mathbf{C} \to \mathbf{C}$ by the equation

$$f(z) = \sum_{j=0}^{\infty} D^j h(O)(j!)^{-1} z^j.$$

Using the hypothesis that $h \in \mathscr{H}_n(0)$ and Lemma A, we easily obtain $f \in \mathscr{E}(0)$. Also, since h is given in the whole of $\mathbf{R}^n$ by its multiple Taylor series about O (see e.g. Brelot [5, Appendix]), we have in particular

$$f(x) = h(x, 0,\ldots, 0) \qquad (x \in \mathbf{R}), \tag{1}$$

so that

$$f(z) = O(|z|^p)$$

as $|z| \to \infty$ through integer values. Hence, by Theorem A, $f \in \mathscr{P}_{[p]}$, and therefore, by (1), the restriction of h to E^1 is a polynomial of degree at most $[p]$ in x_1.

It remains to prove the theorem in the cases where $2 \leqslant m \leqslant n - 1$. In preparation for the proof of these cases, we quote some further lemmas.

LEMMA B. *If $h \in \mathscr{H}(0)$ and $Y \in R^n$, then the function*

$$X \rightsquigarrow h(X + Y) \qquad (X \in \mathbf{R}^n)$$

belongs to $\mathscr{H}(0)$.

This is a simple consequence of the maximum principle.

LEMMA C. *If $P \in \mathscr{P}_{n,q}$, then there exists a harmonic element H of $\mathscr{P}_{n,q}$ such that $H = P$ on E^{n-1}.*

This is well-known and follows, for example, from [2, Lemma 7]. Finally, we recall a weak form of [3, Theorem 1] (see also [7] and [10]).

LEMMA D. *If $h \in \mathscr{H}_m(0)$ and $h = 0$ on I^m, then $h = 0$ on E^m.*

Suppose now that $n \geqslant 3$, $2 \leqslant m \leqslant n - 1$ and that h satisfies the hypotheses of Theorem 1. By Lemma B, if $Y \in I^m$, then the function given by (2) satisfies the hypotheses of the theorem. A fortiori, this function satisfies the hypotheses with $m = 1$, so that, by the case already proved, its restriction to E^1 is a polynomial of degree at most $[p]$, i.e., the restriction of h to the line

$$\{X \in E^m : x_i = y_i \ (2 \leqslant i \leqslant m)\}$$

is a polynomial of degree at most $[p]$. Generalizing this argument in a trivial way, we see that h satisfies the hypotheses of Lemma 1 with $q = [p]$. Hence there is a polynomial P in $\mathbf{R}^n$ such that $h = P$ on I^m. By Lemma C, there is a harmonic polynomial H in $\mathbf{R}^n$ such that $H = P$ on E^{n-1}, and therefore $H = h$ on I^m. Since $H - h \in \mathscr{H}_n(0)$, Lemma D now gives $H = h$ on E^m. Hence the restriction of h to E^m is a polynomial in $x_1, \ldots, x_m$. It remains to prove that the degree of this polynomial is at most $[p]$. Let its degree be s and let h_s be its homogeneous part of degree s. Then there exist a non-degenerate semi-infinite cone C of vertex O in $\mathbf{R}^n$ and a positive number λ such that

$$|h(X)| \geqslant \tfrac{1}{2} |h_s(X)| \geqslant \lambda |X|^s \qquad (X \in C \cap E^m, \lambda |X| > 1).$$

Since C contains infinitely many points of I^m, we must have $s \leqslant p$, else we contradict the hypotheses of the theorem.

To show that the theorem fails if $h \in \mathscr{H}_n(\alpha)$, where $\alpha > 0$, it is enough to observe that the function

$$X \rightsquigarrow \cosh \alpha x_1 \sin \alpha x_n$$

belongs to $\mathscr{H}_n(\alpha)$ and vanishes on E^{n-1} and therefore on I^m for each $m = 1,..., n-1$.

4. Proofs of Theorems 2 and 3

To prove Theorem 2 we shall need the following case of a result of Brelot and Choquet [6, Theorem 6] (or see [1, Theorem 4]).

Lemma E. *If $P \in \mathscr{P}_{n,q}$ and $Z \in \mathbf{R}^n \backslash E^{n-1}$, then there exists a harmonic element H of $\mathscr{P}_{n,q}$ such that $H = P$ on $E^{n-1} + \{O, Z\}$.*

We shall also need the following weak form of [3; Theorem 2] (see also [7] and [10]).

Lemma F. *Let $Z_1 = (0,..., 0, 1)$. If $h \in \mathscr{H}_n(0)$ and $h = 0$ on $E^{n-1} + \{O, Z_1\}$, then $h \equiv 0$.*

Now suppose that h satisfies the hypotheses of Theorem 2. By Theorem 1, the restriction of h to E^{n-1} is a polynomial P_1, say, of degree at most $[p]$ in $x_1,..., x_{n-1}$. By Lemma B, we can use a translation argument to deduce similarly that the restriction of h to $E^{n-1} + \{Z_1\}$ is a polynomial P_2, say, of degree at most $[p]$ in $x_1,..., x_{n-1}$. Now define an element P of $\mathscr{P}_{n,[p]+1}$ by the equation

$$P(x_1,..., x_n) = (a - x_n)\, a^{-1}P_1(x_1,..., x_{n-1}) + x_n a^{-1}P_2(x_1,..., x_{n-1}),$$

where a is the nth coordinate of Z. Then the restriction of P to E^{n-1} is P_1 and the restriction of P to $E_{n-1} + \{Z\}$ is P_2. Hence $P = h$ on $E^{n-1} + \{O, Z\}$. By Lemma E, there is a harmonic element H of $\mathscr{P}_{n,[p]+1}$ such that $H = P = h$ on $E^{n-1} + \{O, Z\}$. Now define a function h^* in $\mathbf{R}^n$ by the equation

$$h^*(X) = (H - h)(aX).$$

Then, clearly, $h^* \in \mathscr{H}_n(0)$ and $h^* = 0$ on $E^{n-1} + \{O, Z_1\}$. By Lemma F, $h^* \equiv 0$, and hence $h = H$.

To show that the degree $[p]+1$ cannot be improved, observe that the function

$$X \rightsquigarrow \operatorname{Im}(x_1 + ix_n)^{[p]+1} \qquad (X \in \mathbf{R}^n)$$

is a harmonic polynomial of degree exactly $[p]+1$ and satisfies the hypotheses of the theorem.

Now suppose that h satisfies the hypotheses of Theorem 3. By Theorem 1, the restriction of h to E^{n-1} is a polynomial P_3, say, of degree at most $[p]$ in $x_1, ..., x_{n-1}$. Since $\partial h/\partial x_n \in \mathscr{H}_n(0)$ [3, Lemma 3], Lemma B allows us to use a translation argument to obtain that the restriction of $\partial h/\partial x_n$ to E^{n-1} is a polynomial P_4, say, of degree at most $[p']$ in $x_1, ..., x_{n-1}$. Define a function H in $\mathbf{R}^n$ by the equation

$$H(X) = \sum_{j=0}^{[\frac{1}{2}p]} (-1)^j((2j)!)^{-1}\, x_n^{2j} \Delta^j P_3(x_1, ..., x_{n-1})$$

$$+ \sum_{j=0}^{[\frac{1}{2}p']} (-1)^j((2j+1)!)^{-1}\, x_n^{2j+1} \Delta^j P_4(x_1, ..., x_{n-1}),$$

where Δ^j is the jth iterated Laplacian in the variables $x_1, ..., x_{n-1}$. It is easy to show by computation that H is a harmonic element of $\mathscr{P}_{n,p^*}$ and that $H = h$ and $\partial H/\partial x_n = \partial h/\partial x_n$ on E^{n-1} (see [2, Lemmas 7 and 8]). A simple argument involving the reflection principle shows that $H - h$ vanishes together with all its partial derivatives on E^{n-1} and hence that $h = H$.

To show that the degree p^* cannot be improved observe that the function

$$X \rightsquigarrow \operatorname{Re}(x_1 + ix_n)^{[p]} + \operatorname{Im}(x_1 + ix_n)^{[p']+1} \qquad (X \in \mathbf{R}^n)$$

is a harmonic polynomial of degree exactly p^* and satisfies the hypotheses of Theorem 3.

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